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OPTIMUM TECHNICAL CHANGE IN AN AGGREGATIVE MODEL OF ECONOMIC GROWTH*

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IN THIS PAPER we are interested in formulating a model of economic growth in which an advancement in the state of technological knowledge is achieved only by engaging scarce resources in some positive quantities, and in analyzing the pattern of the allocation of scarce resources that results in an optimum growth.

Our discussion is carried out in terms of the aggregative model of economic growth, recently introduced by Solow [6, 7] and Swan [8]. The economy is visualized as consisting of two factors of production, labor and capital, which are combined to produce a homogeneous output; any part of the output may be either instantaneously consumed or accumulated as capital stock. The state of technological knowledge existing at each moment of time t is summarized by an aggregate production function, $Y(t) = F(K(t), L_P(t); t)$, which uniquely determines the annual output, $Y(t)$, in terms of the existing capital stock, $K(t)$, and the quantity of labor employed in material production, $L_P(t)$, at time t. Any change in technological knowledge is described by a shift in the aggregate production function. To make the analysis simpler, we shall assume that all changes in technological knowledge are embodied in labor, and that the improvement in labor efficiency does not depend upon the amount of capital to be employed. The aggregate production function at each moment of time, t, then may be written as

$$(1) \qquad Y(t) = F[K(t), A(t)L_P(t)] ,$$

where the state of technological knowledge at time t is represented by the efficiency in labor $A(t)$.[2]

It is assumed that various activities in the form of education, health, construction and maintenance of public goods, etc., which result in an improvement in labor efficiency, $A(t)$, are put together as one sector,

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[2] Technological changes described by the shift of form (1) in the aggregate production function are neutral in the Harrod definition introduced in [1]; see [9].

to be referred to as the educational sector.[3] We postulate that the educational sector employs labor only, and that the impact of activities in the educational sector is uniformly diffused over the whole economy. The rate of improvement in labor efficiency, $\dot{A}(t)/A(t)$, then, may be assumed to be determined by the ratio of labor employed in the educational sector, $L_E(t)$, over the total labor force, $L(t)$,

$$(2) \qquad \dot{A}(t)/A(t) = \phi[L_E(t)/L(t)] .$$

It is assumed that the larger the improvement in labor efficiency, the higher the proportion of labor force employed in the educational sector, with nonincreasing marginal returns, namely,

$$(3) \qquad \phi'(s) \geqq 0, \ \phi''(s) \leqq 0, \ \text{for all } 0 \leqq s \leqq 1 .$$

The available labor apply, $L(t)$, at each moment of time t is assumed to grow at a constant rate n, and to be inelastically supplied, namely,

$$(4) \qquad L_P(t) + L_E(t) \leqq L(t) ,$$

$$(5) \qquad \dot{L}(t)/L(t) = n .$$

The rate of capital accumulation, on the other hand, is determined by the quantity of the annual output to be set aside for investment. Let $I(t)$ and $C(t)$ be respectively the annual rates of aggregate investment and consumption. Then we have

$$(6) \qquad I(t) + C(t) \leqq Y(t) , \qquad I(t), \ C(t) \geqq 0 ,$$

and

$$(7) \qquad \dot{K}(t)/K(t) = I(t) - \mu K(t) ,$$

where μ stands for the rate of depreciation of capital.

The stock of capital, $K(0)$, and labor efficiency, $A(0)$, at the beginning are given as data, together with the pattern of population growth, $L(t)$. The time path of the economy is then uniquely determined when we specify the allocation of labor between the educational and productive sectors, $L_E(t)$ and $L_P(t)$, and the division of the annual output, $Y(t)$, between consumption and investment, $C(t)$ and $I(t)$, at each moment of time.

We are now interested in the problem of finding and characterizing

[3] A number of studies, in particular by Professor Theodore W. Schultz [3, 4], have been made recently on the implications of the improvement in the quality of labor upon the pattern of economic growth. The model presented below has been intended primarily to serve as a basis for discussions on the economic effects of education.

the time path of the optimum economy with respect to the social welfare criterion in terms of the discounted sum of consumption per capita.[4] Let the rate of discount, δ, be given and remain constant, independent of the level of consumption per capita. Then the problem is to find a time path of the economy over which the discounted sum of consumption per capita,

$$(8) \qquad \int_0^\infty \frac{C(t)}{L(t)} e^{-\delta t} dt ,$$

is maximized among all feasible paths resulting from the given initial capital stock, $K(0)$, and labor efficiency, $A(0)$.

It will be assumed in the following that production processes underlying the aggregate production function, $F(K, L)$, are subject to constant returns to scale and to a diminishing marginal rate of substitution between labor and capital. Let output per capita, $y = Y/L$, be related to the capital-labor ratio, $k = K/L$, by the function $y = f(k)$, namely,

$$f(k) = F(K, L)/L = F(k, 1) .$$

Then, $f(k)$ is continuously twice-differentiable, and

$$(9) \qquad f(k) > 0, f'(k) > 0, f''(k) < 0, \quad \text{for all } k > 0 ,$$

$$(10) \qquad f(0) = 0, f(\infty) = \infty ,$$

$$(11) \qquad f'(0) = \infty, f'(\infty) = 0 .$$

We shall assume that

$$(12) \qquad \phi(1) < \delta < \phi(0) + \phi'(0) ,$$

so that quantity (7) is finite for all feasible paths. If $\delta \leqq \phi(1)$, quantity (7) will be indefinitely increased by allocating all available labor force to the educational sector for a sufficiently long period of time. If $\delta \geqq \phi(0) + \phi'(0)$, then it will be seen that, by using a method similar to the one presented below, optimum growth will be obtained by allocating all available labor to the productive sector.

Let us define

$$y(t) = \frac{Y(t)}{L(t)} = \text{output per capita},$$

$$k(t) = \frac{K(t)}{L(t)} = \text{aggregate capital-labor ratio},$$

[4] See T. N. Srinivasan [5] and H. Uzawa [10]. Since labor growth is assumed to be exogenous, our social welfare criterion, with proper change in the discount rate, may also be regarded as based upon aggregate consumption.

$$u(t) = \frac{L_P(t)}{L(t)} = \text{labor allocation to the productive sector,}$$

$$s(t) = \frac{Z(t)}{Y(t)} = \text{investment ratio.}$$

Then our problem is reduced to the following:

Maximize

$$\int_0^\infty (1 - s(t))y(t)e^{-\delta t}dt \, , \tag{13}$$

subject to the restraints

$$\dot{k}(t) = s(t)y(t) - \lambda k(t) \, , \tag{14}$$

$$\dot{A}(t) = A(t)\phi(1 - u(t)) \, , \tag{15}$$

where

$$y(t) = A(t)u(t)f\left(\frac{k(t)}{A(t)u(t)}\right) , \tag{16}$$

$$0 \leqq s(t), u(t) \leqq 1 \, , \tag{17}$$

and $\delta, \lambda = n + \mu, k(0) = K(0)/L(0), A(0)$ are given constants, and $u(t), s(t)$ are piecewise continuous.

The probem thus formulated may be solved by Pontryagin's Maximum Principle, as described in L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko [2]. It is concerned in general with solving problems of the following type:

Given $n+1$ continuously differentiable functions $f^0(x, u), f^1(x, u), \cdots, f^n(x, u)$ defined for n-dimensional vectors $x = (x^1, \cdots, x^n)$ and r-dimensional vectors $u = (u^1, \cdots, u^r)$, find a trajectory $x(t)$ and control $u(t), 0 \leqq t < \infty$, that maximizes the integral

$$J = \int_0^\infty f^0(x(t), u(t))dt \, ,$$

subject to the constraints

$$\dot{x}^i(t) = f^i(x(t), u(t)) \, , \qquad i = 1, \cdots, n \, ,$$
$$u(t) \in U \, , \quad \text{for all } t \, ,$$

where U is a given set of r-vectors (to be referred to as the control region), $x(t)$ is piecewise continuous, and $x(0)$ takes a pre-assigned value.

Pontryagin's Maximum Principle [2, (Theorem 7, p. 69, Theorem 17, pp. 193–94)]: In order that the admissible control $u(t)$ and the corresponding trajectory $x(t)$, $0 \leqq t < \infty$, yield a solution of the optimal problem above, it is necessary that there exist n continuous functions $q^1(t), \cdots, q^n(t)$, such that the following conditions are satisfied:

(a) $$\dot{q}^i(t) = -\frac{\partial H}{\partial x^i}, \qquad i = 1, \cdots, n;$$

(b) $$H(q(t), x(t), u(t)) = \max_{u \in U} H(q(t), x(t), u), \qquad \text{for all } t;$$

(c) $$\lim_{t \to \infty} q^i(t) = 0,$$

where the Hamiltonian $H(q, x, u)$ is defined by

$$H(q, x, u) = f^0(x, u) + q^1 f^1(x, u) + \cdots + q^n f^n(x, u).$$

In our present problem, the Hamiltonian is given by

$$H(q, v, k, A, u, s, t)$$

(18) $$= \left[(1 - s)Auf\left(\frac{k}{Au}\right) + q\left(sAuf\left(\frac{k}{Au}\right) - \lambda k\right) + vA\phi(1 - u)\right]e^{-\delta t}.$$

Applying Pontryagin's Maximum Principle, we then derive

LEMMA 1. *If a time path $(k(t), A(t), u(t), s(t))$, $t \geqq 0$, is optimal, then there exist continuous functions, $q(t)$, $v(t)$, such that*

(19) $$\dot{k}(t) = s(t)A(t)u(t)f\left(\frac{k(t)}{A(t)u(t)}\right) - \lambda k(t),$$

with initial condition $k(0) = K(0)/L(0)$,

(20) $$\dot{A}(t) = A(t)\phi(1 - u(t)),$$

with initial condition $A(0)$,

(22) $$\dot{v}(t) = [\delta - \phi(1 - u(t))]v(t) - p(t)u(t)\left[f\left(\frac{k(t)}{A(t)u(t)}\right) - \frac{k(t)}{A(t)u(t)} f'\left(\frac{k(t)}{A(t)u(t)}\right)\right],$$

(23) $u(t)$ *maximizes* $p(t)uf\left(\frac{k(t)}{A(t)u}\right) + v(t)\phi(1 - u)$,

subject to $0 \leqq u \leqq 1$,

(24) $s(t)$ *maximizes* $(1 - s) + sq(t)$,

subject to $0 \leqq s \leqq 1$,

(25) $$\lim_{t \to \infty} q(t)e^{-\delta t} = \lim_{t \to \infty} v(t)e^{-\delta t} = 0,$$

where

(26) $$p(t) = \max(1, q(t)).$$

To simplify the necessary conditions in Lemma 1, let us introduce the ratio of capital to labor measured in terms of the efficiency unit as a new variable $x(t)$

$$x(t) = \frac{k(t)}{A(t)} = \frac{K(t)}{A(t)L(t)} .$$

Then the time paths of the economy are completely specified by $(x(t), u(t), s(t))$ instead of $(k(t), A(t), u(t), s(t))$. Also, in view of (3) and (9), the expression $puf(x/u) + v\phi(1 - u)$ is a strictly concave function of u; hence, $u(t)$ satisfying (23) is uniquely determined by solving the first-order condition

$$\text{(27)} \qquad \left[f\left(\frac{x}{u}\right) - \frac{x}{u} f'\left(\frac{x}{u}\right)\right] \leqq \frac{v}{p}\phi'(1-u) ,$$

with strict equality if $0 \leqq u < 1$.

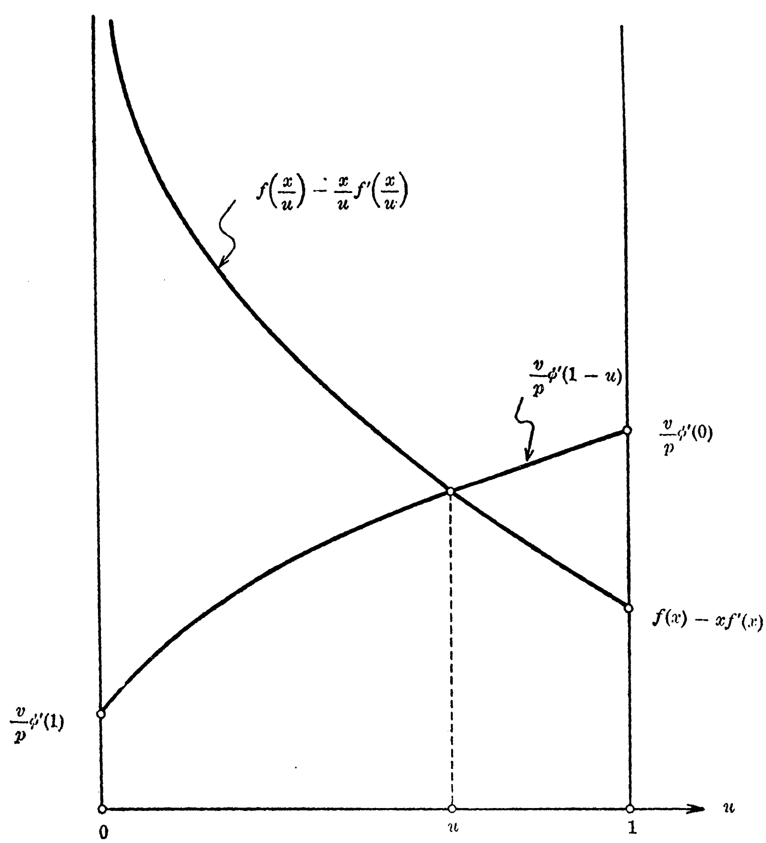

Figure 1

The function $[f(x/u) - (x/u)f'(x/u)]$ is a decreasing function of u, and approaches infinity as u goes to zero, while the value at $u = 1$ is $[f(x) - xf'(x)] > 0$. On the other hand, it is easily seen that $v \geqq 0$ and the function $(v/p)\phi'(1 - u)$ is an increasing function of u, ranging from $v\phi'(1)$ to $(v/p)\phi'(0)$. Hence, the determination of u by condition (27) may be illustrated by Figure 1.

Since the solution u to condition (27) is uniquely determined by x and v/p, we may write

$$u = \psi\left(x, \frac{v}{p}\right).$$

An increase in x is accompanied by an upward shift in the $f(x/u) - (x/u)f'(x/u)$-curve, while the $(v/p)\phi(1 - u)$-curve remains fixed. Hence, as x is increased, u is increased. On the other hand, increasing v/p implies an upward shift in the $(v/p)\phi'(i - u)$-curve, resulting in a decrease in u. We then have

$$\frac{\partial\psi}{\partial x} \geqq 0, \frac{\partial\psi}{\partial(v/p)} \leqq 0, \tag{28}$$

with strict inequality if $u = \psi(x, v/p) < 1$.

Lemma 1 is therefore simplified as follows:

LEMMA 2. *If a time path $(x(t), u(t), s(t))$ is optimal, then there exist continuous functions $q(t)$, $v(t)$ such that*

$$\frac{\dot{x}(t)}{x(t)} = s(t)\frac{f\left(\frac{x(t)}{u(t)}\right)}{\frac{x(t)}{u(t)}} - \lambda - \phi(1 - u(t)), \tag{29}$$

with initial conditon $x(0) = K(0)/A(0)L(0)$,

$$\dot{q}(t) = (\delta + \lambda)q(t) - f'\left(\frac{x(t)}{u(t)}\right)p(t), \tag{30}$$

$$\dot{v}(t) = [\delta + \phi(1 - u(t))]v(t) - p(t)u(t)\left[f\left(\frac{x(t)}{u(t)}\right) - \frac{x(t)}{u(t)}f'\left(\frac{x(t)}{u(t)}\right)\right], \tag{31}$$

$$s(t) = \begin{cases} 0, & if\ q(t) < 1, \\ 1, & if\ q(t) > 1, \end{cases} \tag{32}$$

$$\lim_{t\to\infty} q(t)e^{-\delta t} = 0, \lim_{t\to\infty} v(t)e^{-\delta t} = 0, \tag{33}$$

$$u(t) = \psi\left(x(t), \frac{v(t)}{p(t)}\right), \tag{34}$$

$$p(t) = \max(1, p(t)). \tag{35}$$

Analyzing the conditions in Lemma 2 in detail, let us first consider a particular state of the economy where the differential equations (29)–(31) are singular. It is easily seen that such a case takes place only if $q(t) = 1$. Let (x^*, v^*, u^*, s^*) be obtained by solving the following equations:

$$\phi(1 - u^*) + u^*\phi'(1 - u^*) = \delta,\ 0 < u^* < 1\ , \tag{36}$$

$$f'\left(\frac{x^*}{u^*}\right) = \delta + \lambda\ , \tag{37}$$

$$f\left(\frac{x^*}{u^*}\right) - \frac{x^*}{u^*} f'\left(\frac{x^*}{u^*}\right) = v^*\phi'(1 - u^*)\ , \tag{38}$$

$$s^* \frac{f\left(\frac{x^*}{u^*}\right)}{\frac{x^*}{u^*}} = \lambda + \phi(1 - u^*),\ 0 < s^* < 1\ . \tag{39}$$

The function $\phi(1 - u) + u\phi'(1 - u)$ has a positive derivative, $-u\phi''(1 - u)$, and takes the values from $\phi(1)$ to $\phi(0) + \phi'(0)$ as u moves from zero to unity. Hence, in view of assumption (12), the value u^* satisfying (36) always exists and is uniquely determined. The value x^* then is uniquely obtained by solving equation (37). v^* is determined by substituting the values of u^*, x^* thus obtained into (38). By assumptions (3) and (9), v^* is always positive, and $u^* = \psi(x^*, u^*)$. Equation (39) finally determines the value of s^*. In view of (36)–(38), equation (39) is reduced to

$$s^* = \frac{\lambda + \phi(1 - u^*)}{\lambda + \phi(1 - u^*) + \left(1 + \frac{v^*}{x^*}\right)u^*\phi'(1 - u^*)}\ ;$$

hence, $0 < s^* < 1$.

It is easily seen from equations (36)–(39) that the path defined by $x(t) \equiv x^*$, $v(t) \equiv v^*$, $u(t) \equiv u^*$, $s(t) \equiv s^*$, $q(t) \equiv 1$ satisfies all the conditions (29)–(35) except for the initial condition on $x(0)$. Such a state (x^*, u^*, s^*) will be referred to as the *balanced state* (with respect to the discount rate δ).

The state of the economy is now classified into two phases according to whether $q(t) < 1$ or $q(t) > 1$ for a given amount of time.

Phase I: $q(t) < 1$. In this phase, we have

$$p(t) \equiv 1,\ s(t) \equiv 0\ . \tag{40}$$

Hence, conditions (29)–(32) in Lemma 2 are reduced to the following:

$$\frac{\dot{x}}{x} = -\lambda - \phi(1-u)\,, \tag{41}$$

$$\frac{\dot{q}}{q} = \delta + \lambda - \frac{f'\left(\frac{x}{u}\right)}{q}\,, \tag{42}$$

$$\frac{\dot{v}}{v} \leqq \delta - \phi(1-u) - u\phi'(1-u)\,, \tag{43}$$

with equality whenever $u < 1$, where, for the sake of simplicity, all variables are denoted without explicit reference to time t and $u = \psi(x, v)$; i.e., u satisfies the equality (27) with $p = 1$.

The paths of (x, v) are determined by differential equations (41) and (43) only, independent of the values of q. To investigate the structure of the solutions to the system of differential equations (41) and (43), let us consider the function

$$\beta(x, v) = \delta - [\phi(1-u) + u\phi'(1-u)]\,, \tag{44}$$

where $u = u(x, v)$.

Differentiating $\beta(x, v)$ partially with respect to x, and using (28), we get

$$\frac{\partial\beta}{\partial x} = u\phi''(1-u)\frac{\partial u}{\partial x} < 0\,, \tag{45}$$

and similarly

$$\frac{\partial\beta}{\partial v} = u\phi''(1-u)\frac{\partial u}{\partial v} > 0\,. \tag{46}$$

As x approaches zero, u, also approaches zero; hence, in view of (12), $\beta(x, v) < 0$ if x is sufficiently small. On the other hand, as x becomes larger, the value of u approaches unity; hence again, in view of (12), $\beta(x, v) < 0$ if x is sufficiently large. Similarly, $\beta(x, v) < 0$ if v is sufficiently small, and $\beta(x, v) > 0$ if v is sufficiently large. On the other hand, x is always decreasing in Phase I; hence, the solution paths of (41) and (43) have the pattern illustrated by arrow curves in Figure 2.

The balanced state (x^*, v^*) obviously lies on the $\beta(x, v) = 0$ curve. Among the solution paths of the differential equations (41) and (43), there is one which goes through point (x^*, v^*). Such a path is uniquely determined. Let us define the function $v = v_I(x)$ such that $(x, v_I(x))$ lies on the solution path going through (x^*, v^*).

It is also seen that, for any $x > x^*$, there exists a uniquely determined value $q = q_I(x)$, such that the solution $(x(t), v(t), q(t))$ to the

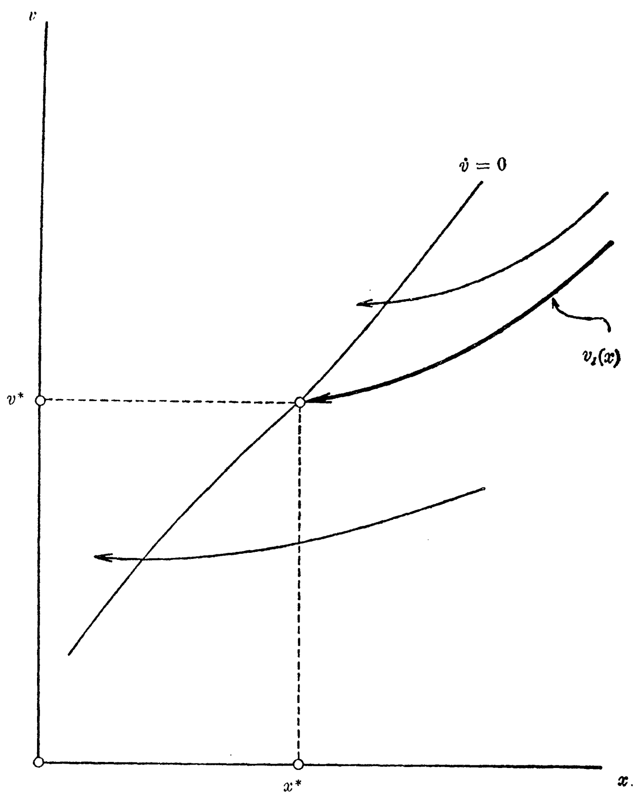

Figure 2

differential equations (41)–(43), with initial condition $(x, v_I(x), q_I(x))$, actually reaches the point $(x^*, v^*, 1)$.

Let the initial capital-labor ratio $x(0) = K(0)/A(0)L(0)$ be greater than the balanced ratio x^*. Then define

$$v_I(0) = v_I(x(0)), q_I(0) = q_I(x(0)) .$$

Let $(x_I(t), V_I(t), q_I(t))$ be the solution to the differential equations (41)–(43), with initial condition $(x(0), v_I(0), q_I(0))$ and t_I be the time when the solution $(x_I(t), v_I(t), q_I(t))$ reaches the point $(x^*, v^*, 1)$; i.e.

$$x_I(t_I) = x^*, v_I(t_I) = v^*, q_I(t_I) = 1 .$$

Now consider the time path $(x(t), v(t), q(t), u(t), s(t))$ defined by

(47) For $0 \leqq t \leqq t_I : x(t) = x_I(t), v(t) = v_I(t), q(t) = q_I(t)$,
$u(t) = \psi(x_I(t), v_I(t)), s(t) = 0$;

For $t > t_I : x(t) = x^*, v(t) = v^*, q(t) = 1$,
$u(t) = u^*, s(t) = s^*$.

The path $(x(t), v(t), q(t), u(t), s(t))$, defined by (47), is the only path satisfying all the necessary conditions (29)–(35) in Lemma 2 when the initial capital-labor ratio $x(0)$ is higher than the balanced ratio x^*.

Phase II: $q(t) > 1$. In this phase, we have

$$p(t) \equiv q(t),\ s(t) \equiv 1 .$$

Hence, conditions (29)–(32) in Lemma 2 are reduced to the following:

$$\frac{\dot{x}}{x} = \frac{f\left(\frac{x}{u}\right)}{\frac{x}{u}} - \lambda - \phi(1-u) , \tag{48}$$

with initial conditions $x(0)$,

$$\frac{\dot{q}}{q} = \delta + \lambda - f'\left(\frac{x}{u}\right) , \tag{49}$$

$$\frac{\dot{w}}{w} \leqq f'\left(\frac{x}{u}\right) - \phi(1-u) - u\phi'(1-u) - \lambda , \tag{50}$$

with equality whenever $u < 1$, where

$$w = \frac{v}{p} \quad \text{and} \quad u = \psi(x, w) .$$

Differential equations (48) and (50) determine the solution paths of (x, w), independent of the values of q. Let us introduce the function $\alpha(x, w)$

$$\alpha(x, w) = \frac{f\left(\frac{x}{u}\right)}{\frac{x}{u}} - \lambda - \phi(1-u) , \tag{51}$$

where $u = \psi(x, w)$; i.e., (27) is satisfied with $v/p = w$.

Differentiating (51) partially with respect to w, we get

$$\frac{\partial \alpha}{\partial w} = \left\{ \frac{f\left(\frac{x}{u}\right) - \frac{x}{u} f'\left(\frac{x}{u}\right)}{x} + \phi'(1-u) \right\} \frac{\partial u}{\partial w} < 0 . \tag{52}$$

On the other hand, let

$$\gamma(x, w) = f'\left(\frac{x}{u}\right) - \phi(1-u) - u\phi'(1-u) - \lambda ; \tag{53}$$

then

$$\frac{\partial \gamma}{\partial x} = f''\left(\frac{x}{u}\right) \frac{\partial\left(\frac{x}{u}\right)}{\partial x} + u\phi''(1-u)\frac{\partial u}{\partial x} < 0 \, . \tag{54}$$

Subtracting (53) from (51), we get

$$\alpha(x, w) - \gamma(x, w) = \frac{f\left(\frac{x}{u}\right) - \frac{x}{u} f'\left(\frac{x}{u}\right)}{\frac{x}{u}} + u\phi'(1-u) > 0 \, . \tag{55}$$

Relations (52), (54) and (55) imply that the solution paths of the differential equations (48) and (50) are those typically illustrated by arrow curves in Figure 3.

Point (x^*, v^*) lies on the $\gamma(x, v) = 0$ curve, and for any $x < x^*$ there exists a uniquely determined $w = w_{II}(x)$ such that the solution to the differential equations (48) and (50) reaches the point (x^*, v^*). This path is indicated by the heavy arrow curve in Figure 3.

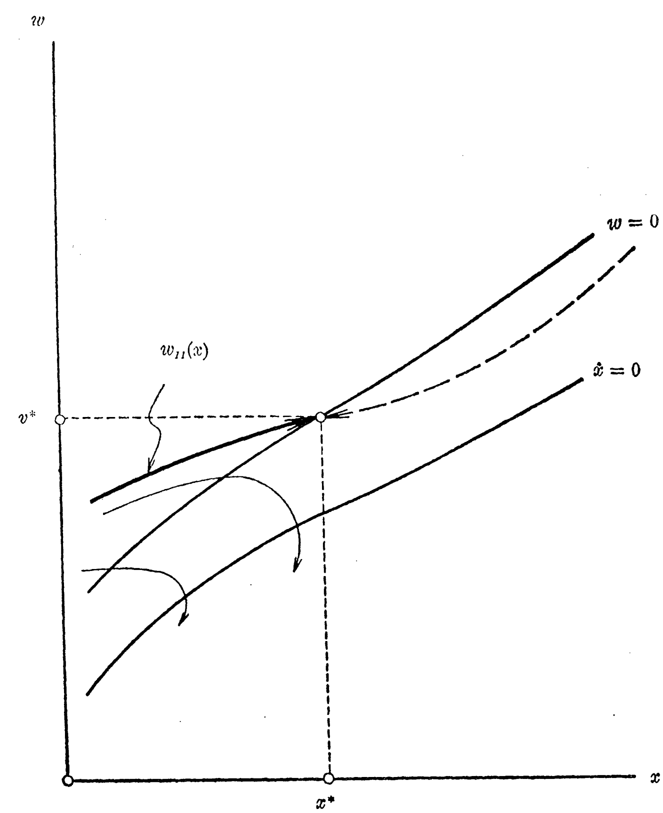

Figure 3

In connection with $x < x^*$, there also exists a uniquely determined value $q = q_{II}(x)$, such that the solution $(x(t), w(t), q(t))$ to the differential equations (48)–(50) with initial condition $(c, w_{II}(x), q_{II}(t))$ actually reaches point (x^*, v^*, q^*).

Let the initial capital-labor ratio $x(0)$ be less than the balanced ratio x^*, and define

$$w_{II}(0) = w_{II}(x(0)), \; q_{II}(0) = q_{II}(x(0)) \; .$$

Let t_{II} be the time when the solution $(x_{II}(t), w_{II}(t), q_{II}(t))$ to the differential equations (48)–(50), with initial condition $(x(0), w_{II}(0), q_{II}(0))$, reaches point $(x^*, v^*, 1)$; i.e.

$$x_{II}(t_{II}) = x^*, \; w_{II}(t_{II}) = v^*, \; q_{II}(t_{II}) = 1 \; .$$

It is easily seen that $q_{II}(t) > 1$, for all $0 \leqq t < t_{II}$, and the time path $(x(t), v(t), q(t), u(t), s(t))$ defined as follows satisfies all necessary conditions in Lemma 2:

$$\text{(56)} \quad \begin{aligned} &\text{For } 0 \leqq t \leqq t_{II}: \\ &\qquad x(t) = x_{II}(t), \; v(t) = w_{II}(t) q_{II}(t), \; q(t) = q_{II}(t) \; , \\ &\qquad u(t) = \psi(x_{II}(t), w_{II}(t)), \; s(t) = 1 \; . \\ &\text{For } t > t_{II}: \\ &\qquad x(t) = x^*, \; v(t) = v^*, \; q(t) = 1 \; , \\ &\qquad u(t) = u^*, \; s(t) = s^* \; . \end{aligned}$$

A simple calculation again shows that the path defined by (56) is the only one which satisfies all the conditions (29)–(35) for the initial $x(0)$ less than x^*; hence it is optimum.

The analysis above may be summarized as follows:[5]

For any rate of discount, δ, satisfying conditions (12), there exists a uniquely balanced capital-labor ratio in terms of the efficiency unit which is obtained by solving equations (36) and (37). If the initial capital-labor ratio in the efficiency unit, $x(0) = K(0)/A(0)L(0)$, is equal to x^*, optimal growth is achieved by allocating labor and annual output such that the rate of increase in labor efficiency, $\dot{A}/A$, equals the rate of increase in the capital-labor ratio, $\dot{k}/k$. The allocation of labor to

[5] One implication of our results is that the pattern of optimal growth is characterized only in terms of the capital-labor ratio in the efficiency unit. Thus, let us consider two economies, one of which has a higher capital-labor ratio and a higher labor efficiency such that the capital-labor ratios in terms of efficiency are identical. The two economies have otherwise identical structures. Then the optimal patterns of labor allocation and investment are identical in the two economies. This is closely related to the proposition advanced by T. W. Schultz in [4, (2-3)].

the productive sector, u^*, is determined by relation (36), while the investment ratio, s^*, is given by (39).

If the initial capital-labor ratio in the efficiency unit, $x(0)$, is greater than the balanced ratio, x^*, then all annual output is consumed until $x(t)$ becomes equal to the balanced ratio, x^*, when the economy is switched to the balanced state. The allocation of labor between the productive and educational sectors during the transient period is precisely determined by (47).

If the initial capital-labor ratio in the efficiency unit, $x(0)$, is less than the balanced ratio, x^*, then all annual output is invested until $x(t)$ reaches the balanced ratio, x^*, when the economy is switched to the balanced state. The precise allocation of labor during the transient period is determined by (56).

The paths described by either (39) or (56) are the only optimum paths.

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