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ON THE TIME CONSISTENCY OF OPTIMAL POLICY IN A MONETARY ECONOMY

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We study the time consistency of optimal monetary policy in a framework akin to the one in [**12**, Ch. 1] but we assume away lump sum taxation—all taxes are distortionary. Our major result is that under perfect foresight (as defined in [**8**, **23**]) optimal monetary policy is bound to be time inconsistent. The paper is closely related to the previous works of Auernheimer [**2**], and Kydland and Prescott [**15**].

1. INTRODUCTION

THE CENTRAL OBJECTIVE of this paper is to discuss the time consistency of Ramsey–Friedman optimal policy (i.e., one that maximizes a sum of instantaneous utilities, where the latter depend on consumption and real monetary balances). The main ingredients of the model are that individuals are *rational*, as defined in [**8**, **23**], and that the issuance or absorption of money is socially costly. The last element distinguishes the present analysis from that in Friedman [**12**], where it is assumed that the quantity of money can be costlessly controlled by resorting to lump-sum taxation, but it makes our model similar in spirit to the one analyzed in Phelps [**17**].

The time-consistency issue is by no means a new one in economics. Strotz [**25**] appears to be the first one to have raised it in relation to an individual consumer in a paper that inspired several other contributions (see, e.g., Hammond [**13**]). Loosely speaking, inconsistency arises in those papers because the individual's taste changes over time. More recently, however, Kydland and Prescott [**15**] (see also [**21**] for more examples and a survey of the literature) have discovered a family of models exhibiting time inconsistency where the source of the problem lies in the technology (particularly the Government's fiscal technology) and in the assumption that people hold rational expectations. Although they briefly touch upon a monetary economy, the central results of their remarkable paper are given in a context where money plays no essential role.

In the monetary literature, Auernheimer [**2**] appears to be the first one to have noticed that time inconsistency could arise if the government attempts to maximize the revenue from money creation.[2] However, the main thrust of his highly perceptive paper has to do with the determination of optimal policy under constraints (like "honest government" rules) that precluded the existence of time inconsistency. The latter was further examined in [**7**, esp. Section 3] in terms of a slight variation of Auernheimer's model. Utilizing the new concepts of

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[2] In a sense, then, Auernheimer's paper is a forerunner of the literature surveyed by Prescott [**21**].

rationality (perfect foresight there) that had been recently expounded by Sargent and Wallace [**23**], I was able to show that the policy that maximizes discounted revenue of money creation was bound to be time inconsistent. However, two important questions were left unanswered there, namely, (i) does there exist an optimal policy (time consistent or not), and (ii) is time inconsistency a direct consequence of a disharmony between the government's and individuals' objectives (as could be the case in [**2**], and it certainly is in [**7**])?

The first question is a rather important one because it could possibly happen that the very nature of monetary models with rational expectations prevented the existence of an optimal policy, a fact that would obviously make the time inconsistency results devoid of any meaning. The second question is also of some interest because the results would cease to be so worrisome and surprising if they simply sprang from the government's attempt to "cheat" the private sector.

In this paper we will give a positive answer to the first question and a negative one to the second. Existence will be discussed in terms of a Cagan–Sargent–Wallace [**2**, **5**, **23**] monetary model. The latter and the time-inconsistency results are presented in Section 2, while the proof of existence is relegated to Appendix 1.

On the other hand, the issue pertaining to the second question above is analyzed in terms of a Sidrauski-type economy, recently studied in a perfect-foresight context by Brock [**4**] and Calvo [**6**]. We will show that time inconsistency of optimal policy is bound to arise even when the government attempts to maximize the welfare of the "representative" individual (or family). Since the basic difference between this model and that considered by Friedman [**12**, Ch. 1] is our assumption that lump-sum taxation is not a feasible policy tool, our result also shows that the time-consistency of Friedman's optimum quantity of money (OQM) rule is strongly dependent on the availability of that kind of policy. Due to the cumbersome technicalities inherent to this case the proofs are sketched out in Appendix 2.

Section 3 is devoted to a verbal discussion of the time-inconsistency issue and to a brief elaboration on some possible solutions.

The paper is closed in Section 4 with a discussion of possible extensions.

2. MODEL AND RESULTS

We will now postulate a very simple model of a monetary economy with perfect foresight which incorporates the central ingredients necessary for generating time inconsistency.

We will assume that the economy produces a homogeneous output, c, which is entirely consumed. Output is a (twice-continuously differentiable) function of net (real) taxes, x, satisfying

(1) $$c = f(x), \quad f''(x) < 0 \text{ for all } x, \quad f(0) > 0 \quad \text{and} \quad f'(0) = 0.$$

Thus, output attains its maximum level when taxes are zero, and there exist $\underline{x} < 0$

and $\bar{x} > 0$ such that

(2) $$f(\underline{x}) = f(\bar{x}) = 0,$$
$$f(x) > 0 \quad \text{for} \quad \underline{x} < x < \bar{x};$$

hence the relevant domain of f is the closed interval $[\underline{x}, \bar{x}]$. Equation (1) is our (admittedly extremely simplified) way of assuming that the types of taxation policies open to the government are all disortionary— the first crucial ingredient of our story.[3]

The demand for real monetary balances, m^d, is given by

(3) $$\ln m^d = -a\pi^*, \quad a > 0,$$

where π^* denotes the expected rate of inflation. This is, of course, the functional form utilized by Cagan [**5**]. The more complicated and certainly economically more satisfactory case where the demand for money is derived from utility maximization is considered in Appendix 2.

An important concept for our discussion is that of a *perfect foresight path*. Here we will follow the approach pioneered by Sargent and Wallace [**23**] (see also [**8**]), according to which a perfect foresight path originated at $t = t_0$ is one along which all markets are cleared, expectations are fulfilled, and, furthermore, the price level is expected to be a positive, continuous,[4] and right-differentiable function of time, from the present to the indefinite future.[5] (In what follows t_0 will sometimes be referred to as the "present time".) Again following [**23**], the fulfillment-of-expectations conditions is taken to mean

(4) $$\pi_t^* = \pi_t \quad \text{for all} \quad t \geqslant t_0,$$

where $t = t_0$ is the present time and π_t is the proportional right-hand derivative of p_t (the price level at t), i.e., $\pi_t \equiv \dot{p}_t^+ / p_t$, where $\dot{p}_t^+$ is the right-hand derivative of p at t.

[3] The assumption that q depends on *net* taxes may be a bit disturbing to the reader. More plausible would be to set $c = \tilde{f}(x_1, x_2)$ where $x_{1,2}$ are gross taxes and subsidies, respectively, and assume $\tilde{f}_i < 0$, $i = 1,2$. In this model it would be inefficient to have $x_i > 0$ for $i = 1,2$. Thus if the government is efficient we have, recalling $x \equiv x_1 - x_2$,

$$c = \tilde{f}(x, 0) \quad \text{if } x \geqslant 0,$$
$$= \tilde{f}(0, -x) \quad \text{if } x \leqslant 0,$$

which is essentially what we asserted in (1).

[4] Continuity of the expected price level path can easily be justified when there is an asset like neoclassical capital as an alternative to money, because an anticipated jump in the price level would lead individuals to try to shift away from money into capital or vice versa, a situation that would be inconsistent with market clearing (see [**8**]). In the present context, where there is no capital accumulation, the continuity condition is less compelling. However, if we assume that output is storable, a reasoning like the one given above (and in [**8**]) could again be used to argue that expected price jumps would not be consistent with market clearing. See also footnote 7.

[5] There is another condition on convergence to a steady state which will be introduced later on in connection with the uniqueness issue (see Appendix 1).

The market-clearing condition implies

(5) $$m_t^d = \frac{M_t}{p_t} \equiv m_t, \quad \text{for all} \quad t \geqslant t_0$$

where M_t is the nominal stock of money at t.[6]

An important characteristic of the above definition of perfect foresight—and one that the reader must firmly keep in mind to avoid being mystified by the ensuing developments—is that it puts no constraint on the *present* price level, i.e., on p_{t_0}, in relation to its past values, i.e., in relation to p_t for $t < t_0$. Thus although we constrained p to be continuous and right-differentiable for all $t \geqslant t_0$, we have imposed no regularity conditions like left-continuity or differentiability at t_0. Consequently, p could "take a jump" at t_0 with respect to its past values. Thus, under perfect foresight, the past behavior of the price level imposes no constraint on its equilibrium future values. See [**23**] for a clear and convincing exposition of this principle.

We assume that government debt consists entirely of money, and we will examine paths where government consumption is identically equal to zero (see Section 4 where the implications of relaxing this assumption are briefly discussed). Under these assumptions we must have

(6) $$M_t = M_{t_0} - \int_{t_0}^{t} p_v x_v \, dv, \quad t \geqslant t_0 \quad \text{(government's budget constraint).}$$

By (2) it is natural to constrain x_v (*all* $v \geqslant t_0$) to be in the interval $[\underline{x}, \bar{x}]$. For analytical convenience we will also constrain x_v to be right-continuous and piece-wise continuous on the interval $[t_0, \infty]$. Therefore, since by definition, in a perfect foresight path, p_v is continuous for all $v \geqslant t_0$, it follows from (6) that *in a perfect foresight path originated* at $t = t_0$ *M_t is continuous and piece-wise differentiable for* $t \geqslant t_0$. The latter coupled with (3)–(5) implies that π_t is continuous for all $t \geqslant t_0$ and, hence, that *in a perfect foresight path originated at* $t = t_0$ *p_t is continuously differentiable for all* $t \geqslant t_0$.

Since

(7) $$\frac{\dot{M}_t}{p_t} \equiv \dot{m}_t + \pi_t m_t$$

and, by (3)–(6),

(8a) $$\pi_t = -\frac{\ln m_t}{a}$$ (implied by clearing of the money market and self-fulfilling expectations),

(8b) $$\frac{\dot{M}_t}{p_t} = -x_t$$ (implied by government's budget constraint),

[6] The money market is the only one for which equilibrium will be required in the present section because it is the only one market for which supply and demand functions are going to be modelled (this is also the case in [**7**, **23**]). In Appendix 2, however, the equilibrium condition will be imposed on both the money and the output market.

we have that in a perfect foresight path originated at $t = t_0$

$$x_t = \frac{m_t \ln m_t}{a} - \dot{m}_t \tag{9}$$

at all points $t \geqslant t_0$ such that x_t is continuous; thus (9) holds piece-wisely on the interval $[t_0, \infty)$. This, and the previous assumption on x_t, also shows that *along a perfect foresight path originated at* $t = t_0$, $\dot{m}_t$ *is right-continuous and piece-wise continuous on the interval* $[t_0, \infty)$.

Suppose now that the government's objective function is given by

$$\int_{t_0}^{\infty} [u(c_t) + v(m_t)] e^{-\delta(t-t_0)}\, dt \tag{10}$$

(if the integral fails to converge, paths are ordered by Weiszacker's overtaking principle [**26**]). This is the type of objective function studied by Friedman [**12**, Ch. 1] and more recently by Phelps [**17**]. Separability is assumed in order to simplify the mathematical derivations but it is in no way essential for the central results. We also assume

(11a) *$u(c)$ and $v(m)$ are defined on* $(0, \infty)$ *and twice continuously differentiable and strictly concave*;

(11b) $\lim_{c \to 0} u(c) = \lim_{m \to 0} v(m) = -\infty$;

(11c) $u' > 0$ *for all* $c > 0$, *and there exists* $m = m^F$ *such that* $v'(m^F) = 0$.

Assumption (11a) is just a "regularity" condition that will simplify the mathematics; (11b) will serve the purpose of ruling out "corners" (i.e., $c = 0$ or $m = 0$) along intervals of the optimal plan (as defined below). The assumption on u in (11c) is perfectly standard; the assumption on the existence of m^F, on the other hand, could be dispensed with but at the cost of not being able to define an optimum quantity of money (OQM). I decided to keep it given the importance that such a concept has in related issues of monetary theory (see [**12**]).

Let us define

$$h(x) = u(f(x)). \tag{12}$$

By (1), (2), and (11) we have

(13a) *$h(x)$ is defined on* $(\underline{x}, \bar{x})$, *it is twice-continuously differentiable and strictly concave*;

(13b) $\lim_{x \to \underline{x}} h(x) = \lim_{x \to \bar{x}} h(x) = -\infty$;

(13c) $h'(x) \gtreqless 0 \quad as \quad x \lesseqgtr 0.$

We are now prepared to analyze the problem of maximizing (10) along perfect foresight paths. In view of our previous discussion the latter is equivalent to maximizing (10) subject to (9) and $\underline{x} < x_t < \bar{x}$ for all t. Furthermore, m_{t_0} is free to take any value in the interval $(0, \infty)$—because p_{t_0} is free to take any positive value; m_t, as shown before, is constrained to be continuous on $[t_0, \infty)$, and $\dot{m}_t$ to be piece-wise continuous over the same interval. Taking (9) and (12) into account, the government's problem can therefore be stated as follows:

$$(14) \qquad \max_{\{m\}} \int_{t_0}^{\infty} \left[h\left(\frac{m_t \ln m_t}{a} - \dot{m}_t \right) + v(m_t) \right] e^{-\delta(t-t_0)}\, dt$$

subject to

$$(15a) \qquad 0 < m_{t_0},$$

(15b) m_t *continuous and* $\dot{m}_t$ *right-continuous and piece-wise continuous on* $[t_0, \infty)$,

$$(15c) \qquad \underline{x} < \frac{m_t \ln m_t}{a} - \dot{m}_t < \bar{x} \textit{ at all points where } \dot{m} \textit{ is defined.}$$

Sufficient conditions for existence and uniqueness of a solution are given in the Appendix. In what follows we will take existence and, to simplify the exposition, also uniqueness for granted; the optimal m-path when calculated at t_0 is indicated $m^*(t; t_0)$, $t > t_0$.

The first question that we have to solve is whether the government has enough tools to generate the optimal m-path. Remember that in this model we are letting the price level be determined by "market forces", so the government cannot directly choose the values of m; it can only do so indirectly by operating through people's expectations. We will show in Appendix 1 that $m^*(t; t_0)$ could be generated by announcing the path of M which is *associated* with the optimal plan; the latter is easily calculated because along the optimal plan we should have

$$(16) \qquad p_t = \frac{M_t}{m^*(t; t_0)};$$

thus, recalling (6)

$$(17) \qquad M_t = M_{t_0} - \int_{t_0}^{t} \frac{M_v}{m^*(v; t_0)} x_v\, dv.$$

This formula can be used to calculate the associated M-path given the optimal x-path since, recalling (9), we have

$$(18) \qquad x_t = \frac{m^*(t; t_0) \ln m^*(t; t_0)}{a} - \dot{m}^*(t; t_0),$$

for all t where $\dot{m}^*$ is defined.

We will now turn to the time-inconsistency issue. Formally we will say that there is time inconsistency if for some $\theta > 0$ we have

$$(19) \qquad m^*(t; t_0) \neq m^*(t; t_0 + \theta)$$

for some $t \geqslant t_0 + \theta$. In other words, there is time inconsistency if the optimal value of m at $t > t_0$ when calculated at time t_0 is not optimal from the vantage point of some future time (before t). The implications of time inconsistency for optimal monetary policy will be discussed in the next section.

Since the optimum problem at t_0 is identical to the one faced at $t_0 + \theta$ it is quite clear from (19) that in order to rule out time inconsistency we must have

(20) $$m^*(t; t_0) = \text{some constant};$$

we will now argue that it is very unlikely that (20) is satisfied. Without loss of generality, we will carry the discussion for the case $t_0 = 0$.

Clearly, an optimal *stationary* policy is equivalent to the Golden Rule for this model and, recalling (14), should therefore maximize

(21) $$h\left(\frac{m \ln m}{a}\right) + v(m).$$

By (11) and (13) there is an interior solution (i.e., with $0 < m$), the first order condition being

(22) $$h'\left(\frac{m \ln m}{a}\right)\frac{1 + \ln m}{a} + v'(m) = 0.$$

In order to rule out time inconsistency we must have $m^*(t; 0) \equiv \bar{m}$ where $\bar{m}$ is some m maximizing (21). For if the latter does not hold it is clear that $m^*(t; 0)$ would not be constant as required by (20).

We show in the Appendix that a necessary condition for an optimal policy at $t = 0$ is

(23) $$h'(x_0) = 0$$

or, by (13c), $x_0 = 0$ (i.e., zero taxes at time zero). For our present discussion it will be enough to prove (23) for m constant optimal policies since, as argued above, those are the only candidates if time consistency is going to prevail.

By (18), if $m_t \equiv \bar{m}$ then

(24) $$x_t = \frac{\bar{m} \ln \bar{m}}{a} = \bar{x} \text{ for all } t.$$

Suppose, contradicting (23), $h'(\bar{x}) \neq 0$. Consider a new path where

(25) $$\begin{aligned} m_t &= \bar{m} + A\left(1 - \frac{t}{t_1}\right)^2, \quad 0 \leqslant t < t_1, \\ &= \bar{m}, \quad t \geqslant t_1, \end{aligned}$$

for some constant A and time $t_1 > 0$. Thus, for $A > 0$, for example, the new path looks like the one in Figure 1.

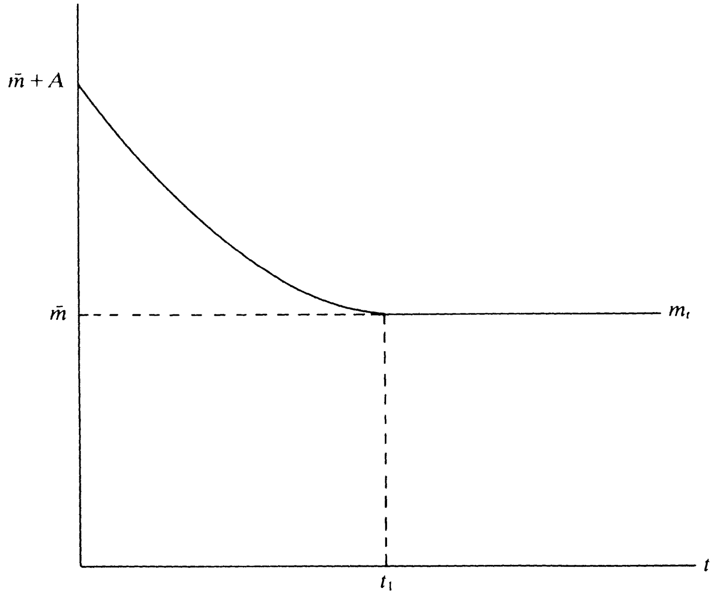

FIGURE 1

Furthermore, by (9), its associated x_t satisfies

$$(26)\qquad x_t=\frac{m_t\ln m_t}{a}+2A\frac{t_1-t}{t_1^2},\quad 0\leqslant t<t_1,$$
$$=\bar{x}\quad t\geqslant t_1.$$

Hence, recalling (10), the difference between the government's utility at $t=0$ derived from the new path and the one having $m_t=\bar{m}$ for all t is

$$(27)\qquad \int_0^{t_1}\left\{h\left(\frac{m_t\ln m_t}{a}+2A\frac{t_1-t}{t_1^2}\right)+v(m_t)-h(\bar{x})-v(\bar{m})\right\}e^{-\delta t}\,dt.$$

Differentiating (27) with respect to A at $A=0$ we get, recalling (25),

$$(28)\qquad \int_0^{t_1}\left\{\left[h'(\bar{x})\frac{1+\ln\bar{m}}{a}+v'(\bar{m})\right]\left(1-\frac{t}{t_1}\right)^2+h'(\bar{x})2\frac{t_1-t}{t_1^2}\right\}e^{-\delta t}\,dt.$$

As t_1 goes to zero the integral of the first term inside the curly brackets in (28) times $e^{-\delta t}$ converges to zero because $(1-(t/t_1))$ is a number between zero and one. On the other hand

$$(29)\qquad 2\frac{h'(\bar{x})}{t_1^2}\int_0^{t_1}(t_1-t)\,e^{-\delta t}\,dt$$

is seen to converge to $h'(\bar{x})$ as t_1 tends to zero by twice applying L'Hôpital rule. Hence, for sufficiently small t_1, (29) has the sign of $h'(\bar{x})$ implying, therefore, that

(27) can be made positive by setting A sufficiently close to zero and

(30) $A \gtreqless 0$ as $h'(\bar{x}) \gtreqless 0$.

In words, we have found that if $h'(\bar{x}) \neq 0$ it *is possible to increase the government's utility by increasing (decreasing)* m_0 *over* m if $h'(\bar{x}) > 0(>0)$, i.e., recalling (13c) if $\bar{x} < 0(>0)$. Consequently, if a Golden Rule path were to be optimal we should have $h'(\bar{x}) = 0$ or, equivalently, $\bar{x} = 0$. This proves (23) for optimal stationary paths.

In view of the above discussion and combining (22) and (23), we can then conclude that a necessary condition for an optimal policy to be time consistent is that $m_t = \bar{m}$ for all t and

(31a) $\bar{x} = 0$,

(31b) $v'(\bar{m}) = 0$.

But, by (9), (31a) requires ln $\bar{m} = 0$ and hence $\bar{m} = 1$. Then the fulfillment of (31) is dependent upon

(32) $v'(1) = 0$ (or, recalling (11c), $m^F = 1$),

a condition that cannot be derived from the assumptions of the model. We can then assert that optimal policies will not generally be time consistent—the exception possibly being when $v'(1) = 0$, i.e., $m^F = 1$.

Equation (31b) has a strong resemblance to the OQM full-liquidity condition, except that in the standard analysis (see [**12**, Ch. 1]) the government maximizes the utility of the representative individual and here we have made no reference to the latter. With that caveat in mind, however, the value of m satisfying (31b) may still be called the OQM because it gives the satiation level of real monetary balances *as seen by the government*. In these terms, our results can be stated in the following suggestive manner: *A time-consistent optimal policy should generate the OQM and be associated with zero distortionary taxes and/or subsidies at every point in time.* The problem in the present economy arises because the two conditions cannot in general be simultaneously fulfilled: the OQM requires in general imposition of distortionary taxes or subsidies.

To summarize, we have argued (a complete proof is in Appendix 1) that it is optimal to set taxes equal to zero at the beginning of the plan given that taxes (or subsidies) reduce output, and that one is free to choose the initial condition for real monetary balances. The latter is a direct consequence of the important fact that in perfect foresight paths, as here defined, the "present" price level is free to take a jump with respect to its past values, and that this can only happen at the beginning of the plan because the future expected path of prices is constrained to be continuous, or to put it in an economically more meaningful way, because after the future economic policy is announced today our rational individuals will realize that future price jumps will be inconsistent with market clearing (recall footnote 4) and would, therefore, not expect them. Thus at the start of a plan, the government has, so to say, one degree of freedom that is lost for future points in time (when seen from the present time) due to the nature of perfect foresight expectations. When the future arrives, however, expectations of the (then) present held in the (then) past lose all their relevance. The government can therefore recover the lost degree of freedom when replanning in the future. As a consequence if future plans are going to be consistent with that in the present, optimal taxes should be zero at all points in time, and thus, by (6) nominal money supply should also remain constant over time; but, as shown by (31) and (32), such a plan would be "dominated" by another where taxes are constant over time and different from zero except, perhaps, in the special case where the OQM is attained at a specific value ($=1$ in our model).[7]

On the other hand, if, contrary to our assumptions, it is possible to resort to lump sum (nondistortionary) taxation (as in [**12**, Ch. 1]), the optimal policy at t_0 would clearly be to generate the OQM for all $t \geqslant t_0$ because fiscal policy could be arranged in such a way that c is always at its maximum level; hence inconsistency would not arise.

2. DISCUSSION OF THE RESULTS AND OF SOME SOLUTIONS TO THE TIME INCONSISTENCY DILEMMA

With perfect foresight—or more generally with rational expectations—expectations are formed on the basis of the structure of the economy and information about the government's intended policy. Thus, at any time t an opportunistic planner could try to deceive people so as to induce them to, for instance, hold the OQM and produce the maximum output at t. But such a strategy is probably worthless after individuals learn the trick because it would not be rational for them to trust what the government says. The time inconsistency of optimal policy disclosed in the previous section also implies that the planner will change the announced policy in the future and, hence, run into the same kind of difficulties indicated above. The reasons, however, are more subtle. The concept of optimality at time t_0 that is defined in Section 2 requires that the optimal policy at t_0 maximize government's utility under the assumption that *there is no cheating whatsoever*. In fact, from a formal point of view this optimality concept is identical to the one discussed by, for example, Arrow and Kurz [**1**, Ch. 5–8] who did not encounter any case of time inconsistency.

Time inconsistency is in our model due to (a) the nature of the demand for money and (b) the fact that its creation implies in general the use of distortionary taxation. The first element enters the picture because the amount of *real* monetary balances that people are willing to hold today, say, depends on the rate of inflation between today and tomorrow, but the balances held tomorrow are not; they are only a function of the rate of inflation between tomorrow and the

[7] Lucas (unpublished notes) has proven essentially the same result in the context of a discrete-time finite-horizon version of this model, which indicates that one could considerably relax the continuity condition discussed in footnote 4 and still be able to show time inconsistency. Another discrete-time example is given in [**7**].

ensuing day. Notice that this would not be so if we were talking of capital instead of money because its (real) quantity at t_0, say, would be determined by decisions taken in the past.[8]

Consequently, continuing with the discrete-time story, from the vantage point of today changes in the expected price level of tomorrow in relation to today's will affect the real stock of money today and will, therefore, have to be taken into account by an optimizing government; when tomorrow arrives, however, those changes are irrelevant. This opens the door for time inconsistency. But it is not sufficient for it, because we have also shown that no time inconsistency would arise (in our model) if it is possible to lump-sum tax or subsidize (this highlights the essentiality of point (b) above).

Readers familiar with Bellman's *Optimality Principle* (see [**1**, Ch. 2]) would probably not be entirely convinced by the above remarks.[9] For if a planner with unchanging tastes and full information has any reason to depart at time t_0+h, $h>0$, from a plan that looked optimal at t_0, then it would seem to follow that the plan could not have been optimal at time t_0 either, given that maximizing a sum from t_0 to $+\infty$ requires that the plan also maximizes the sum from t_0+h *to* $+\infty$. This is, of course, impeccable reasoning *if the constraints facing the planner at time* t_0 *are the same as those at time* t_0+h; but such is not the case, because in a world of rational expectations the planner at t_0 who discloses the nature of his plan can only consider surprise-free paths on the interval $[t_0, \infty)$; the same planner at t_0+h is again constrained to surprise-free plans on $[t_0+h, \infty)$ but he is in no way bound to only choose among those plans that would be considered surprise-free (or fully anticipated) when coupled with the history from t_0 to t_0+h, and seen from the standpoint of time t_0. Time inconsistency arises because it is optimal to exploit that element of surprise, and the latter, by the very nature of the rationality hypothesis, cannot be planned in advance.

Time inconsistency of optimal policy in a rational world has devastating implications: it devoids the optimum problem studied in Section 2 of any meaning whatsoever because rational individuals realize that it will be optimal for the government in the future to modify the policies which are optimal from the standpoint of today. As a consequence, any proposal for solving that problem must entail a revision of the government's objectives or a constraint on the set of feasible policies. Here we will discuss the second type of alternative. The former alternative is explored in the next section. However, before getting into that we wish to clarify a question related to point (ii) mentioned in the introduction—namely, does time inconsistency arise because the government does not maximize the utility of the representative individual?

A negative answer to the question will be given in Appendix 2 where we show that time inconsistency is also generally true in a world of identical and infinitely-

[8] This argument carries over to the case of heterogeneous capital as long as the planner's utility depends on the stock of each one of them, and not on some arbitrary aggregate.

[9] I am very thankful to my friend and colleague Ronald Findlay for extremely helpful discussions on this topic.

lived individuals or families with utility function

$$\int_{t_0}^{\infty} [u(c_t)+v(m_t)]e^{-\delta(t-t_0)}\,dt \tag{33}$$

and where the demand for money and consumption at every point in time are derived from utility maximization under perfect foresight. This will leave no doubt, we hope, that time inconsistency is not necessarily a consequence of a disharmony between public and private interests or of the Samuelson–Diamond imperfections that may arise when individuals have a finite life (see [**24**, **10**]).

Let us now turn to consider some possible solutions to the time-inconsistency dilemma. It is clear that no inconsistency arises if the government optimizes at t_0, say, and abides by the dictates of that policy for all $t \geqslant t_0$; so one possible proposal could be constraining the government to do just that for a given t_0. The determination of t_0, however, does not appear to be a trivial matter. Notice that all planners after t_0 would be forced to non-optimize even when it is, in principle, feasible for them to revise the value of t_0. In a realistic situation, of course, such revisions will breed distrust on the part of the private sector and might, hence, be counter-productive. But since that can only be determined after more is known about the individuals' response to those policy changes, the expedient of setting a fixed date for optimization appears at best to be incomplete.

Auernheimer [**2**] suggested setting up a rule by which money supply is adjusted so as to prevent the present price level from jumping with respect to its past values, which obviously solves the time-inconsistency problem. However, although in Auernheimer's model such a rule could possibly be argued to be the one "an honest government" (his words) would like to pursue, its moral appeal is greatly diminished in the present context because, as pointed out above, we encounter time inconsistency even when the government attempts to maximize the welfare of the representative individual, that is to say, in a context where there is not a shade of malevolence or dishonesty at play. Consequently, this suggestion is subject to the same criticisms of the previous criterion since, in the absence of a moral argument, Auernheimer's solution is essentially the same as the latter.

Another solution which is consistent with much of the literature on the maximization of revenue from money creation (see Friedman [**11**], Marty [**16**]) is the Golden Rule—i.e., the maximization of steady-state utility. This solution will certainly work in the simple model of the previous Section but it is meaningless in more realistic settings where due to the presence of "state variables" (e.g., different types of durable capital) the economy is simply not at a steady state.

Finally there is the Phelps-Pollak solution to time inconsistency (Phelps and Pollak [**19**], Phelps [**18**]) in which, roughly speaking, the present government maximizes discounted utility taking as given the policies of future governments. However, although time inconsistency is avoided by construction it has the serious drawback that solutions are in general Pareto inefficient, uniqueness cannot be easily ensured,[10] and, rather disturbingly, there may be two solutions where one is strictly Pareto superior to the other (i.e., every government would be better off in one of them compared to the other; see Phelps [**18**]).

4. CLOSING REMARKS

1. The model studied in this paper can be enriched and modified in several directions without changing the central time inconsistency results. This is so, in particular, if we allow for positive government consumption. As a matter of fact, one can show that if the latter enters into the government's objective function, and income affects the demand for money, time inconsistency could arise even when money supply is constrained to be constant over time.

2. In the model of Section 2, time consistency would prevail, however, if instead of (10) we postulated the government's utility to be

$$\inf_{t \geq t_0} \{u(c_t)+v(m_t)\}. \tag{34}$$

One can easily show that the Golden Rule of the model in Section 2 maximizes (34) (but also that many other paths do). The objective function given by (34) bears a strong resemblance with the *maximum principle* (see [**9**, **19**, **20**, **22**]) and it would be consistent with it if, for instance, $u(c_t)+v(m_t)$ could be thought of as the utility of generation t (or of government t). It remains to be investigated, however, whether an objective function like (34) or, more generally, the maximin principle leads to time consistent optimal policies in more realistic cases.

3. Although our discussion has centered around economies where individuals are rational, the reader should not be led to conclude that these types of difficulties are inherent in only those cases. To be sure, any economy where individuals are sensitive to the announcement of future policies has, in principle, the seeds of time inconsistency. As an example, consider the case where the demand for money is a function of μ_{t_0+1}). At t_0+1 the optimal policy would again expansion of money supply at $t+1(\equiv \mu_{t+1})$. Assuming, as in Section 2, that money issuance (absorption) is distortionary, it is clear that if a monetary policy maximizes (10) it will have $\mu_t=0$ for $t_0 \leq t < t_0+1$ since the latter maximizes output in that interval and has no effect on the demand for money (given that the demand for money is a function of μ_{t_0+1}). At t_0+1 the optimal policy would again call for setting $\mu_t=0$, for $t_0+1 \leq t < t_0+2$, and so on. Thus, if there is a time-consistent policy it should have $\mu_t \equiv 0$ for all t. But, in the same fashion of

[10] In the model of Section 2, for instance, one can show that every steady state is a Phelps–Pollak solution. However, see [**7**, **15**] for an example where the solutions are unique. In order to realize how inefficient this type of solution could be, the reader is referred to [**7**] where it is shown that in a context where the government's objective is to maximize the revenue from money creation, Phelps–Pollak solution calls for setting the rate of expansion of the money supply at its maximum *feasible* level.

Section 2, one can show that, except in one exceptional case, the $\mu_t \equiv 0$ policy will be dominated by another one where $\mu_t \equiv$ a nonzero constant.

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APPENDIX 1

By (14), the Hamiltonian of the maximum problem stated in Section 2 when $t_0 = 0$ is

$$H = h\left(\frac{m_t \ln m_t}{a} - \dot{m}_t\right) + v(m_t) + \lambda_t \dot{m}_t, \quad t \geq 0, \tag{A1}$$

where λ is the costate variable of m. (From here on time subscripts will be deleted unless they are strictly necessary.) Maximization of H with respect to $\dot{m}$ yields

$$h'(x) = \lambda; \tag{A2}$$

thus, given (13), (A2) has a unique solution and we can set

$$x = \tilde{x}(\lambda), \quad \tilde{x}'(\lambda) < 0, \quad \tilde{x}(0) = 0. \tag{A3}$$

Let us define

$$\bar{H} = \max\{H: \dot{m} \text{ any number}\}; \tag{A.4}$$

then

$$\bar{H} = h(\tilde{x}(\lambda)) + v(m) + \lambda\left[\frac{m \ln m}{a} - \tilde{x}(\lambda)\right] \tag{A.5}$$

and hence

$$\frac{\partial^2 \bar{H}}{\partial m^2} = v''(m) + \lambda \frac{1}{am} \tag{A.6}$$

which is negative if $\lambda \leq 0$. This fact will be instrumental below for proving existence and uniqueness.

Applying the techniques of optimal control (see, e.g., Arrow and Kurz [**1**]) we get

$$\dot{\lambda} = -v'(m) - \lambda\left[\frac{1 + \ln m}{a} - \delta\right]. \tag{A.7}$$

Also, by (9) and (A3),

$$\dot{m} = \frac{m \ln m}{a} - \tilde{x}(\lambda). \tag{A.8}$$

(A.7 and 8) must be satisfied in an optimal solution; their phase diagram is depicted in Figure 2 under the assumption that $(1/a) > \delta > 0$ and the OQM m (indicated by m^F) is larger than one. By (8a), the latter is equivalent to saying that the rate of inflation associated with the OQM is negative. Furthermore, we have assumed that there is a steady state like $(\hat{\lambda}, \hat{m})$ in the diagram. A sufficient condition for the latter—given all previous assumptions—can be shown to be

$$f\left(\frac{m^F \ln m^F}{a}\right) > 0, \tag{A.9}$$

which simply says that it is feasible to generate the OQM.[11] Since m_0 is free to take any nonnegative

[11] The point $\underset{\sim}{m}$ in Figure 2 indicates the value of m such that $1 + \ln m = a\delta$; since we made the realistic assumption that $a\delta < 1$ it follows that $\underset{\sim}{m} < 1$.

value and by (11b) $m_0=0$ could not be optimal, the following transversality condition at the origin must hold:

(A.10) $\quad \lambda_0=0.$

Thus (23) follows immediately from (A.2) and (A.10).

The path starting at m_0^* in Figure 2 and converging to $(\hat{\lambda}, \hat{m})$ satisfies all the above necessary conditions and also

(A.11) $\quad \lim_{t\to\infty} m_t\lambda_t e^{-\delta t}=0.$

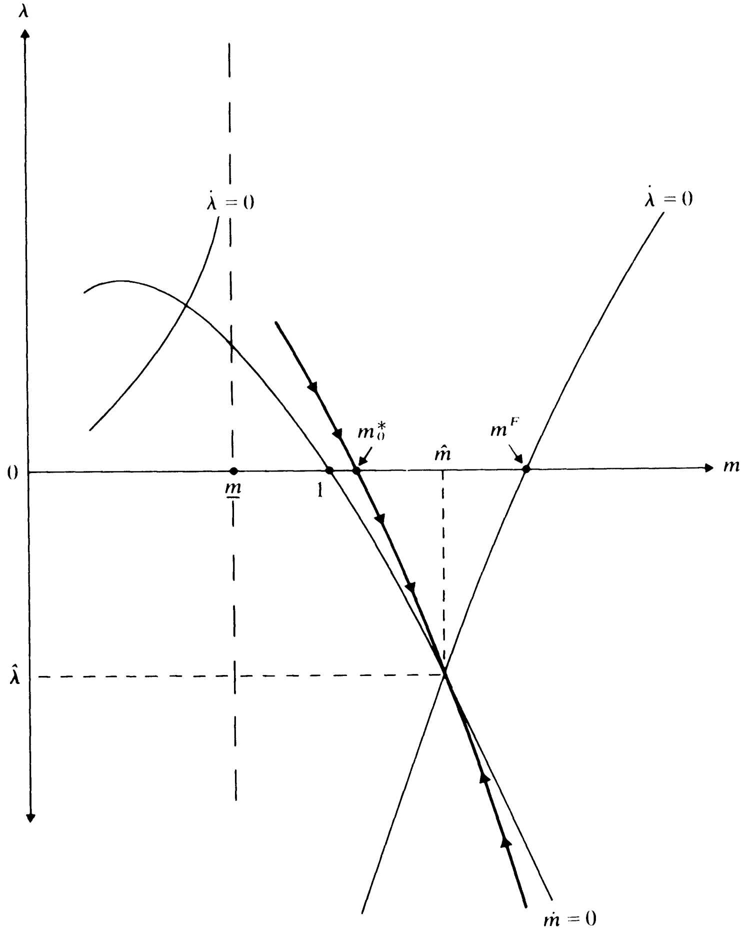

FIGURE 2

Hence, since on that path $\lambda_t \leqslant 0$ for all t, then (A.6) holds for all t which allows us to apply the sufficiency theorem for optimal controls (see Kamien and Schwartz [**14**]) for proving that such a path is in fact optimal at $t=0$. The strict concavity of $\bar{H}$ with respect to m for $\lambda \leqslant 0$ ensures uniqueness.

Notice that by (A.3), (A.10), and Figure 2, optimal x starts at zero and monotonically approaches $x(\hat{\lambda})>0$. Thus along the optimal plan both m and x are monotonically increasing functions of time.

Also, by (8a), optimal π starts at $(-\ln m_0^*/a)<0$ and monotonically decreases towards $(-\ln \hat{m}/a)$. Time inconsistency arises because at any future time the government faces the same maximization problem and will therefore set initial m at the level indicated by m_0^* in Figure 2—implying that the plan which is optimal from the standpoint of time zero will cease to be so in the future.

Now suppose that at time 0 the government announces M according to formula (17) employing the optimal path of m and x to generate it. Clearly

$$\frac{\dot{M}_t}{M_t}=-\frac{\ln m^*(t;0)}{a}+\frac{\dot{m}^*(t;0)}{m^*(t;0)}.$$

Hence

$$\frac{1}{a}\int_t^\infty e^{-(s-t)/a}\left\{\ln M_0+\int_0^s\left[-\frac{\ln m^*(v;0)}{a}+\frac{\dot{m}^*(v;0)}{m^*(v;0)}\right]dv\right\}ds=\frac{1}{a}\int_t^\infty e^{-(s-t)/a}\ln M_s\,ds$$

exists for all $t\geqslant 0$ because m^* converges to $\hat{m}$ (see Figure 2). But the last expression is the natural log of the only perfect foresight path of p generated by the associated M path if all of Sargent and Wallace **[23]** and Calvo **[8]** characterizations of perfect foresight paths are adopted (see Sargent and Wallace [**23**, equation 5]). Thus the perfect foresight path of p is uniquely determined by announcing the M path associated with the optimal plan. It is now an easy matter to check that given that path of prices, equilibrium m follows the m^* path.

In other words, we have given conditions under which there exists a money supply path such that if it is announced at time zero it maximizes the government's utility (at time zero) in an environment where economic agents hold rational expectations. This policy has the characteristic that taxes are zero at the beginning of the plan and monotonically increase towards $\hat{x}>0$. Thus distortions are minimized at time zero and increase with time. (The fact that taxes are zero at the beginning of the plan does not follow from the assumption that $\delta>0$, as one might inadvertently tend to think—it follows instead from the fact that p_0 and thus m_0, is free to jump). Given that government expenditures are kept equal to zero, the positivity of future taxes implies that money supply is a decreasing function of time.

APPENDIX 2

Here we will present a sketch of the proof that time inconsistency is bound to arise in a world of identical families with a utility function like (33) and a government whose aim is to maximize the welfare of the representative individual.

A family takes x_t—and hence net output $f(x_t)$—and π_t as exogenously given and maximizes (33) with respect to $m(\cdot)$ and $c(\cdot)$ subject to the budget constraint

$$\dot{m}_t=f(x_t)-c_t-\pi_t m_t,\quad t\geqslant t_0,$$

an initial stock of real monetary balances. This is a control problem for which the associated Hamiltonian is

(A.12) $$H=u(c)+v(m)+\gamma[f(x)-c-\pi m].$$

Hence, maximization with respect to c yields

(A.13) $$u'(c)=\gamma$$

and we must have

(A.14) $$\dot{\gamma}=-v'(m)+\gamma[\pi+\delta].$$

In a perfect foresight path (5) holds and there is equilibrium in the output market. Hence assuming that output cannot be accumulated and making the number of families equal to one, we have,

(A.15) $$c_t=f(x_t)\quad \text{for all } t,$$

which, in view of (A.13) and (A.14) implies (defining $\mu = \dot{M}/M$)

$$u''(f(x))f'(x)\dot{x}_t = -v'(m) + u'(f(x))\left[\mu - \frac{\dot{m}}{m} + \delta\right] \tag{A.16}$$

or

$$\dot{m} = \frac{m}{u'(f(x))}[-v'(m) + u'(f(x))(\mu + \delta) - u''(f(x))f'(x)\dot{x}]. \tag{A.16)'}$$

On the other hand, it is clear that (8b) is equivalent to $x_t = -\mu_t m_t$; thus

$$\dot{x} = -\dot{\mu}m - \mu\dot{m}. \tag{A.17}$$

Combining (A.16)' and (A.17) we then get

$$\dot{m} = \frac{m}{u'(f(x)) - \mu m u''(f(x))f'(x)}[-v'(m) + u'(f(x))(\mu + \delta) + u''(f(x))f'(x)\dot{\mu}m]; \tag{A.18}$$

hence, recalling that $x = -\mu m$, in perfect foresight equilibrium

$$\dot{m} = \psi(m, \mu, \dot{\mu}) \tag{A.19}$$

where ψ is defined for all $m > 0$, μ and $\dot{\mu}$ except at (m, μ) such that the denominator of the right-hand side of (A.18) vanishes; but, since by (8b) the latter is only a function of μm we could constrain functions f and u such that the denominator of (A.18) never vanishes and, hence, ψ is defined for all $m > 0$, μ and $\dot{\mu}$. We assume such is the case in what follows.

The government maximizes (33) subject to (A.15) and (A.19). In order to apply the techniques of optimal control we will think of m and μ as "state variables" and $\dot{\mu}$ as the "control"; furthermore we are free to choose m_0 and μ_0. The Hamiltonian becomes

$$H = u(f(-\mu m)) + v(m) + \gamma_1\psi(m, \mu, \dot{\mu}) + \gamma_2\dot{\mu} \tag{A.20}$$

where γ_1 and γ_2 are the costate variables of m and μ, respectively.

For basically the same reasons given in the text, if there is a time-consistent optimal policy at $t = 0$, then there must be one with a constant m. Since the transversality conditions at the origin are

$$\gamma_i(t_0) = 0, \quad i = 1, 2, \tag{A.21}$$

one can also argue that there must be a choice of "supporting" costate variables such that

$$\gamma_i(t) = 0 \quad \text{for all } t \text{ and } i = 1, 2. \tag{A.22}$$

But the γ_i's must satisfy

$$\dot{\gamma}_1 = u'f'\mu - v'(m) - \gamma_1\psi_m + \gamma_1\delta, \tag{A.23a}$$

$$\dot{\gamma}_2 = u'f'm - \gamma_1\psi_\mu + \gamma_2\delta. \tag{A.23b}$$

Thus (A.22) and (A.23) imply $f' = 0$ and $v'(m) = 0$. Thus, as in the example discussed in the text, time consistency requires $x = 0$ (no taxes) and the OQM. But, as before, those two conditions will not hold at the same time in general. So also here optimal policies are time inconsistent in general.

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